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ON A CLASS OF DEGENERATE EXTREMAL GRAPH PROBLEMS

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Given a class $\mathcal{L}$ of (so called "forbidden") graphs, ex $(n, \mathcal{L})$ denotes the maximum number of edges a graph G^n of order n can have without containing subgraphs from $\mathcal{L}$. If $\mathcal{L}$ contains bipartite graphs, then ex $(n, \mathcal{L})=O(n^{2-c})$ for some $c>0$, and the above problem is called *degenerate*. One important degenerate extremal problem is the case when C_{2k}, a cycle of $2k$ vertices, is forbidden. According to a theorem of P. Erdős, generalized by A. J. Bondy and M. Simonovits [3$_2$, ex $(n, \{C_{2k}\})=O(n^{1+1/k})$. In this paper we shall generalize this result and investigate some related questions.

0. Notation

We shall restrict our consideration to ordinary graphs without loops and multiple edges. If G is a graph, $e(G)$, $v(G)$, and $\chi(G)$ will denote the number of edges, vertices, and the chromatic number respectively. As an alternative way to indicate the number of vertices, we shall use superscripts. More precisely, we agree that a capital letter with a superscript always denotes a graph and the superscript always denotes the number of vertices in this graph. K_p, C_p, $K_{p,q}$ and $C_{k,t}$ denote the complete p-graph, the cycle of length p, the complete bipartite graph with p and q vertices in its classes, and the (k, t)-theta-graph, respectively. (The (k, t)-theta-graph is obtained by joining two vertices x and y by t vertex-independent paths of length k.) Some further notations will be introduced later.

1. Introduction

Given a family of so called *forbidden* graphs, one can ask:

Problem 1. What is the maximum number of edges a graphs G^n (of n vertices) can have without containing any $L\in\mathcal{L}$ as a subgraph?

This number will be denoted by ex $(n, \mathcal{L})$, or by ex (n, L), if $\mathcal{L}$ consists of one graph L. EX $(n, \mathcal{L})$ denotes the family of those graphs for which the maxi-

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mum is attained. These graphs will be called extremal for $\mathscr{L}$. Here we shall consider only ordinary graphs without loops or multiple edges, although analogous questions are also investigated for multigraphs, diagraphs and hypergraphs, (see [2], [5], [17]). To be quite precise, the expression "containing an $L\in\mathscr{L}$" always means "containing a subgraph L' isomorphic to some $L\in\mathscr{L}$".

Let us define the *subchromatic number* $p(\mathscr{L})$ by

$$p(\mathscr{L}) = \min_{L\in\mathscr{L}} \chi(L)-1. \tag{1}$$

The value of $\mathrm{ex}\,(n,\mathscr{L})$ is asymptotically given by a theorem of Erdős and Simonovits [12] stating that

$$\mathrm{ex}\,(n,\mathscr{L}) = \left(1-\frac{1}{p}\right)\binom{n}{2}+o(n^2). \tag{2}$$

Further, they proved in [6], [7], and [16], that if $S^n\in\mathrm{EX}\,(n,\mathscr{L})$ then it can be obtained from a complete p-partite graph $K_{n_1,n_2,\dots,n_p}$ (where n_i is the size of the i^{th} class) by changing $o(n^2)$ edges only. The classes are approximately of the same size: $n_i=\frac{n}{p}+o(n)$. The minimum degree $\underline{d}(S^n)=\left(1-\frac{1}{p}+o(1)\right)n$. The meaning of of these results is that the structure of the extremal graphs depends above all on the subchromatic number of $\mathscr{L}$, and only very loosely on the actual members of it. Therefore it is very similar to the structure of the extremal graph $T_{n,p}$ in Turán's famous theorem [19], where $\mathscr{L}=K_{p+1}$ is frobidden. Thus the structure of extremal graphs for $p\geqq 2$ can be regarded as asymptotically sufficiently well described.

The situation suddenly changes as we turn to the case $p=1$. Now the first term of (2) vanishes and the error term $o(n^2)$ is much larger than the actual value of $\mathrm{ex}\,(n,\mathscr{L})$. This case will be called *degenerate*. One of the first results in this field is the

Kővári—T. Sós—Turán theorem [15]. $\mathrm{ex}\,(n, K_{p,q})=O(n^{2-1/p})$. Some finite geometric constructions of E. Klein [8] and W. G. Brown [4] show that the above theorem is sharp for $p=1, 2, 3, q\leqq p,$ and it is conjectured that it is sharp for every p and $q\geqq p$. For $K_{2,2}$ and $K_{3,2}$ even the multiplicative constant of $n^{3/2}$ is known, [4], [11]. Here we are primarily interested in the case of even cycles. (For odd cycles $\mathrm{ex}\,(n, C_{2k+1})=[n^2/4]$ if $n>n_0(k)$.) An unpublished theorem of P. Erdős asserts that

$$\mathrm{ex}\,(n, C_{2k}) = O(n^{1+\frac{1}{k}}). \tag{3}$$

(Later Bondy and Simonovits [3] published a proof of a generalization of this result, see also [2].) Analyzing their proof, Bondy and Simonovits arrived at the conjecture that if $C_{k,t}$ denotes the (k,t)-theta-graph, that is, the graph of order $2+t(k-1)$ obtained by joining two vertices x and y by t vertex-independent paths of length k, then

$$\mathrm{ex}\,(n, C_{k,t}) = O(n^{1+\frac{1}{k}}), \tag{4}$$

too. (The length of a path is the number of its edges.) Obviously, this would generalize (3), since $C_{k,2}=C_{2k}$. However, the method of Bondy and Simonovits was not applicable to $C_{k,t}$.

In this paper we shall prove a general "recursion theorem" which will yield, among others the above conjecture:

Theorem 1. *Given two integers k and t, there exists a constant $c_{k,t}>0$, such that*

$$\mathrm{ex}\,(n, C_{k,t}) \leqq c_{k,t} n^{1+\frac{1}{k}}.$$

As to the sharpness of Theorem 1, we think that even (3) is sharp for every $k \geqq 2$, which would immediately imply the sharpness of Theorem 1. Some finite geometrical constructions of Singleton and Benson [18], [1] and from [4] show that (3), and therefore Theorem 1 as well, is sharp for $k=2$, 3 and 5. It may seem strange, but the case $k=4$ is still unsettled. Recall that (using random graphs) Erdős and Rényi [10] proved that if $v(L)=v$, $e(L)=e$ and $c_L>0$ is a sufficiently small constant, then *the probability that a random graph with n labelled vertices and $[c_L n^{2-v/e}]$ edges contains L is less than* 1. Thus

$$\mathrm{ex}\,(n, L) > c_L n^{2-\frac{v}{e}}, \quad (c_L > 0). \tag{5}$$

This yields that

$$\mathrm{ex}\,(n, C_{k,t}) > c'_{k,t} \cdot n^{2-\frac{t(k-1)+2}{tk}} = c_{k,t} n^{1+\frac{1}{k}-\frac{2}{tk}}, \tag{6}$$

for some constant $c'_{k,t}>0$. Here the exponent tends to $1+\frac{1}{k}$ as $t \to \infty$. In this (rather weak) sense Theorem 1 is sharp.

Another consequence of (5) is that the Kőváry—T. Sós—Turán theorem is also sharp in the sense that

$$\mathrm{ex}\,(n, K_{p,q}) > c^*_{p,q} n^{2-\frac{1}{p}-\frac{1}{q}}, \quad (c_{p,q} > 0) \tag{7}$$

and here the exponent tends to $2-\frac{1}{p}$ if p is fixed and $q \to \infty$.

As we have already mentioned, Theorem 1 will be deduced from a much more general result. To formulate this we need a definition.

Definition 1. Let T be an arbitrary bipartite graph with a fixed 2-colouring C using red and blue. Take a vertex w not belonging to T and join it to each red vertex of T by pairwise vertex-independent paths of length $k-1$. The resulting graph will be denoted by $L_k = L_k(T, C)$.

Remark. If T is connected, then it has exactly two 2-colourings with red and blue. If it has k components, the number of colourings is 2^k. Hence we may get 2^k different graphs L_k from the same T.

Examples. Figures 1 and 2 represent the cases $T=C_6$ and $T=K_{2,3}$, respectively. In the second case we get two different L'_ks, $(k=5)$.

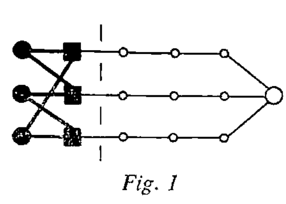

Fig. 1

Fig. 2

Theorem 2. *If T is a tree (or a forest) with a given 2-colouring C by red and blue, then for $L_k = L_k(T, C)$*

$$\mathrm{ex}(n, L_k) = O(n^{1+\frac{1}{k}}). \tag{8}$$

Obviously, if $T = K_{1,t}$ and c colours the t-vertex class of $K_{1,t}$ by red, then $L_k = C_{k,t}$. Hence Theorem 2 implies Theorem 1, consequently it implies (3) as well.

As a matter of fact, we shall prove a slightly more general result. To formulate it we need

Definition 2. Let L be a given bipartite graph with a fixed 2-colouring by red and blue, and colour the first class of a $K_{n,n}$ red, the second one blue. Then $\mathrm{ex}^*(n, L)$ denotes the maximum number of edges a subgraph G^n of $K_{n,n}$ can have without containing *an L whose blue and red vertices are in the blue and red class of $K_{n,n}$*. (Thus, the graph G^n considered here is allowed to contain an L whose red vertices are in the blue class, or, if L is not connected, then it may also happen that the considered subgraph G^n contains an L which has red vertices in both classes of $K_{n,n}$.)

Remark. In principle $\mathrm{ex}^*(n, L)$ may be much larger than $\mathrm{ex}(n, L)$.

Theorem 2. Under the conditions of Theorem 2*

$$\mathrm{ex}(n, L_k) \leqq \mathrm{ex}^*(n, L_k) = O(n^{1+\frac{1}{k}}). \tag{8*}$$

Remark. Let T, C, and k be fixed as in Definition 1, and fix also an odd integer $s \geqq 1$. Let $L_{k,s} = L_{k,s}(T, C)$ denote the graph obtained from T by fixing a new vertex w and joining each red vertex of T to w by a path of length $k-1$ and each blue vertex by a path of length $k+s-1$, so that these paths are vertex-independent and have only their (other) endvertices in common with T (see figure 3). Now, as we shall see, if T is a tree, then

$$\mathrm{ex}^*(n, L_{k,s}) = O(n^{1+\frac{1}{k}}). \tag{9}$$

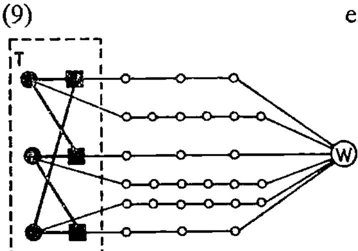

Fig. 3

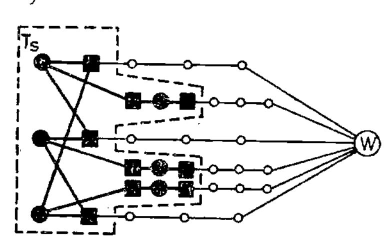

Fig. 4

Since $L_{k,s} \supseteq L_k$, (9) seems to be much stronger than Theorem 2*. However, they are equivalent. To see this, let $y_1, ..., y_q$ denote the blue vertices of T. Let T_s be the graphs obtained by fixing q new vertices $z_1, ..., z_q$ and joining y_i to z_i for $i=1, 2, ..., q$ by vertex-independent paths of length s, where we assume that these paths have no vertices in T but their endvertices y_i. Clearly, T_s is again a tree, therefore we may apply Theorem 2* to T_s and the colouring C' obtained by extending the original colouring C of T onto T_s. It is easily seen that $L_k(T_s, C') \supseteq L_{k,s}(T, C)$. Hence, by Theorem 2*,

$$\text{ex}^*(n, L_{k,s}) \leqq \text{ex}^*\big(n, L_k(T_s, C')\big) = O(n^{1+\frac{1}{k}}). \tag{10}$$

This proves (9). Clearly, the above argument applies also to the case, when for different blue vertices we choose (possibly) different s'es.

The methods of this paper also yield

Theorem 3. *Let T be an arbitrary bipartite graph for which*

$$\text{ex}^*(n, T) = O(n^{2-\alpha}), \tag{11}$$

or some $\alpha \in (0, 1]$. Then for $\beta = \dfrac{\alpha+\alpha^2+\ldots+\alpha^{k-2}}{1+\alpha+\ldots+\alpha^{k-2}}$

$$\text{ex}(n, L_k) \leqq \text{ex}^*(n, L_k) = O(n^{2-\beta}). \tag{12}$$

Theorem 3 is *not* a direct generalization of Theorem 2*, since for $\alpha=1$ it gives only a weaker estimate, namely

$$\text{ex}^*(n, L_k) = O(n^{1+\frac{1}{k-1}}).$$

Probably Theorem 3 could be improved: β could be replaced by $\beta' = \dfrac{\alpha+\alpha^2+\ldots+\alpha^{k-1}}{1+\alpha+\ldots+\alpha^{k-1}}$, and in this case we would have a proper generalization of Theorem 2*. Here we give only a sketch of the proof of Theorem 3 and we are going to return to a more detailed discussion of the corresponding general problem in a next paper.

2. The "Blowing up" method

Denote by $\underline{d}(G)$ the minimum degree in G. We shall use

Proposition 1. (Erdős). *Every graph G^n contains a bipartite subgraph H^n such that $\underline{d}(H^n) \geqq \frac{1}{2}\underline{d}(G^n)$.*

Proposition 1 follows immedaitely from the following, sharper

Proposition 2. (Erdős). *Let H be a bipartite subgraph of G having the maximum number of edges, and let $d_H'(x)$ and $d_G(x)$ denote the degrees in the corresponding graphs. Then for every vertex x of G $d_H(x) \geqq \frac{1}{2} d_G(x)$.*

We shall also need the following "regularization" lemma.

Proposition 3. (Folklore) *Every graph G^n contains a subgraph H^n such that $\underline{d}(H^m) \geqq e(G^n)/n$.*

The proof of Proposition 3 is trivial by induction on n: for $n=1,2$ it is true, and if we have a G^n, then either all vertices have degree $\geqq e(G^n)/n$, or we can delete one, say x, with $d(x)<e(G^n)/n$. Clearly, for $G^{n-1}=G^n-x$,

$$\frac{e(G^{n-1})}{n-1}>\frac{e(G^n)}{n}.$$

Thus we may apply the induction hypothesis to G^{n-1}.

Lemma 1. *Let L be a fixed bipartite graph with a fixed colouring (in red and blue). Let T be a tree and M be the graph obtained from vertex-disjoint copies of L and T by identifying a vertex x of L and a vertex y of T. Then*

$$\mathrm{ex}^*(h,L)\leqq \mathrm{ex}^*(h,M)\leqq \mathrm{ex}^*(h,L)+v(M)h.$$

In particular, if L is just a vertex, we have

$$\mathrm{ex}^*(h,T)\leqq v(T)h$$

Proof. We may assume that L has no isolated vertices. Let G^h be a bipartite graph with the colour-classes V_1 and V_2, and assume that

$$e(G^h)>\mathrm{ex}^*(h,L)+v(M)h.$$

Delete all vertices of degree $<v(M)$. Then the remaining graph G^k contains an L, since

$$e(G^k)\geqq e(G^h)-(h-k)v(M)>\mathrm{ex}^*(h,L)+kv(M).$$

By definition, we may also assume that the red vertices of L are in V_1, the blue ones in V_2. Then we build up an M in G^k as follows. Since T is a tree, we may define a sequence of graphs M_j $(j=0,1,\dots,v(T))$, so that $M_0=L$, $M_{v(T)}=M$ and M_i is obtained from M_{i-1} by adding a new vertex u_i and a new edge (u_i,w_i). (M_i has $v(L)+i$ vertices and w_i is in M_{i-1}). We prove by induction on i that a copy $\tilde{M}_i$ of M_i occurs in G^k. $L=M_0\subseteq G^k$. Assume that $\tilde{M}_{i-1}$ is a copy of M_{i-1} in G^k. Find the $\tilde{w}_i$ corresponding to w_i in M_{i-1}. To get a copy of M_i in G_k we may choose any neighbour u_i of $\tilde{w}_i$ *but* the ones in $\tilde{M}_{i-1}$. Since the degree of w_i is at least $v(M)$ in G_i, we can fix a neighbour $\tilde{u}_i\notin V(\tilde{M}_{i-1})$, thus obtaining a copy $\tilde{M}_i$ of M_i in G^k. ∎

Lemma 2. *Let G^m be a bipartite graph with colour-classes U and V, $|V|\geqq|U|$, and assume that G^m contains no L with its red vertices in U. If $\mathrm{ex}^*(n,L)\leqq Kn^{2-\gamma}$, then*

$$e(G^m)\leqq 4K|V|\cdot|U|^{1-\gamma}. \tag{13}$$

Proof. Partition the vertex-set V into $\left[\frac{|V|}{|U|}\right]$ sets $V_1,\dots,V_q$ of sizes at least $|U|$, at most $2|U|$. Then the subgraph G_j spanned by U and V_j has at most $4K|U|^{2-\gamma}$ edges. Thus

$$e(G^m)\leqq\frac{|V|}{|U|}4K|U|^{2-\gamma}=4K|V|\cdot|U|^{1-\gamma}. \;∎$$

Corollary 1. *Using the notation of Lemma* 2, set $d=e(G^m)/|U|$. *(Clearly, this is the average degree in U.) Then*

$$|V| \geqq \frac{1}{4K} d|U|^{\gamma}. \tag{14}$$

Proof. (14) immediately follows from

$$d|U| = e(G^m) \leqq 4K|U|^{1-\gamma}|V|. \blacksquare$$

The above assertions were more or less trivial. The next lemma is different in the sense that this is where the heart of the proof is hidden.

Lemma 3. *Let T be a bipartite graph with a fixed two-colouring C. Let* $\mathrm{ex}^*(n, T) \leqq Kn^{2-\gamma}$ *for some constants K and* γ. *For a given k assume that the bipartite graph H contains no* $L_k=L_k(T, C)$. *Let u be a vertex of H and* S_i *denote the set of vertices of distance i from u,* $i=1, 2, \ldots, p+1 \leqq k-1$. If $\bigcup_{i \leqq p} S_i$ spans a tree *F in H and each vertex of* S_p *is joined to* S_{p+1} *by at least* $r/2$ *edges, where for* $m=v(L_k)$, $r \geqq (K+m)v(H)^{1-\gamma}$, *then for* $c_p=\frac{1}{2}\left(\frac{1}{-32(K+m)}\right)^p$,

$$|S_{p+1}| \geqq c_p r |S_p|^{\gamma}. \tag{15}$$

Proof. We shall use induction on p. For $p=0$ (15) is trivial, since $|S_0|=1$, $|S_1| \geqq r/2$, and $c_0=1/2$. Assume Lemma 3 for $p-1$. Now we shall prove it for p.

Let $S_1=\{u_1, \ldots, u_R\}$. We shall say that a vertex $w \in S$ is above u_j, if the path joining w to u in the *tree* S goes through u_j. The vertices above u_j and the vertices of S_{p+1} joined to them satisfy the conditions of Lemma 2 with $p-1$. Finally, each vertex of S_p is (still) joined to at least $r/2$ vertices of S_{p+1}.

Hence we can apply the hypothesis to each u_j: if A_j denotes the set of vertices of S above u_j in S_p, $|A_j|=a_j$ and if B_j is the set of vertices in S_{p+1} joined to A_j by an edge, then for $b_j=|B_j|$ we have

$$b_j \geqq c_{p-1} r a_j^{\gamma}. \tag{16}$$

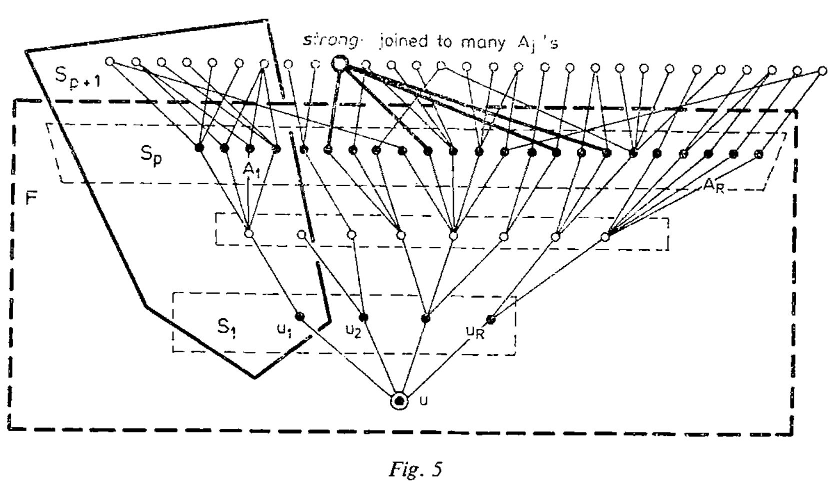

Fig. 5

Here the sets A_j are disjoint, therefore $\sum a_j = |S_p|$. Thus

$$\sum b_j \geqq c_{p-1} r \sum a_j^\gamma \geqq c_{p-1} |S_p|^\gamma \tag{17}$$

(since $0<\gamma\leqq 1$). We would be home if the B_j's were disjoint, but generally they are not. The basic idea below is that either there are many vertices belonging only to a few B_j's and this case is almost the same as if the B_j's were disjoint, or many vertices of S_{p+1} are covered many times and then we can apply Lemma 2, more precisely, its corollary.

(A) First we define the strong vertices and strong edges. A *vertex* $w \in S_{p+1}$ will be called strong, if it belongs to at least $m=v(L_k)$ sets B_j, that is, if w is joined to at least m different sets A_j. An $S_p S_{p+1}$*-edge* whose endvertex in S_{p+1} is strong will also be called *strong*. We assert that if B is the graph spanned by the strong edges, $s=k-p-2$ and T_s is the graph obtained from T by hanging independent s-paths on each red vertex of T, then B *cannot contain a* T_s *whose endvertices are in* S_{p+1}.

Indeed, assume that B contains a T_s with q red vertices and the new endvertices of the corresponding s-paths are $w_1, \dots, w_q \in S_{p+1}$. Since these vertices are strong, we may choose q edges (t_i, w_i) in B so that *the vertices* t_i *belong to* q *different* $A_{i_j} \subseteq S_p$. Join each t_i to u by a p-path, thus we get q vertex-independent paths of length $k-1$ from the red vertices of T to u: we get an $L_k \subseteq H$. This contradiction proves the assertion.

(B) Assume next that

$$e(B) \geqq \frac{r}{8} |S_p|. \tag{18}$$

By Lemma 1, $\mathrm{ex}^*(n, T_s) \leqq (K+m)n^{2-\gamma}$. By Lemma 2 (as in the proof of Corollary 1) we obtain that B has at least $\frac{r}{32(K+m)} |S_p|^\gamma$ vertices in S_{p+1}. Indeed, denote the set of these vertices by V. Then we show that $|V| \geqq |S_p|$, from which, by Corollary 1,

$$|S_{p+1}| \geqq |V| \geqq \frac{1}{(K+m)} \frac{r}{8} |S_p|^\gamma$$

immediately follows. To show that $|V| \geqq |S_p|$, observe that otherwise $v(B) < 2|S_p|$ and $e(B) > \frac{r}{16} v(B) > (K+m) v(B)^{2-\gamma}$. Thus B contains a T_s with its endvertices in S_{p+1}. This was excluded.

(C) Finally, we assume that

$$e(B) < \frac{r}{8} |S_p|. \tag{19}$$

Let W be the (bipartite) graph defined by the weak $S_p S_{p+1}$ edges. There are at least $1/2|S_p|$ vertices in S_p incident to $\geqq r/4$ weak edges, (otherwise there were at least $1/2|S_p|$ vertices in S_p incident to $\geqq r/4$ strong edges, which contradicts (19)).

If $\tilde{A}_j \subseteq A_j$ denotes the set of vertices incident with $\geqq r/4$ weak edges and $\tilde{B}_j$ denotes the set of other endvertices of these weak edges (in S_{p+1}), then a subtree F_j "joining" $\tilde{B}_j$ to u_j enables us to apply the induction hypothesis with $\tilde{a}_j=|\tilde{A}_j|$, $\tilde{b}_j=|\tilde{B}_j|$ and $r/2$:

$$\tilde{b}_j \geqq c_{p-1}\frac{r}{2}\tilde{a}_j^{\gamma},$$

(instead of (16)) and

$$\sum \tilde{b}_j \geqq c_{p-1}\frac{r}{2}(\sum \tilde{a}_j^{\gamma}) \geqq c_{p-1}\frac{r}{2}(\sum \tilde{a}_j)^{\gamma}.$$

Since now we work with *weak* vertices and edges, each vertex of $\cup\tilde{B}_j$ is counted at most m times in $\sum\tilde{b}_j$. Thus

$$|S_{p+1}| \geqq |\cup\tilde{B}_j| \geqq \frac{1}{m}\sum \tilde{b}_j \geqq \frac{1}{m}c_{p-1}\frac{r}{2}(\sum \tilde{a}_j)^{\gamma}.$$

On the other hand, $\sum\tilde{a}_j=|\cup\tilde{A}_j|\geqq 1/2|S_p|$, therefore

$$|S_{p+1}| \geqq c_{p-1}\frac{r}{4m}|S_p|^{\gamma} \geqq c_p r|S_p|^{\gamma}.$$

Corollary 2. Under the conditions of Lemma 3, if $\gamma=1$, then

$$|S_{p+1}| \geqq c_p r|S_p|.$$

Remark. For arbitrary bipartite T we can prove Lemma 3 only if $p+1\leqq k-1$. However, if T is a tree, then T_s is also a tree, and in part (A) of the proof we can easily build up T_s in the graph B (of the strong edges) vertex by vertex, so that each vertex of T_s be in different A_j or in S_{p+1}. Indeed, if we have already fixed a subtree $T^*\subseteq T_s$ in B and we wish to add an edge (u_i, w_i) to it, where u_i must be chosen from S_p, then w_i is strong, therefore u_i may be chosen from at least m different A_j's and we have used up at most $m-1$ of them. This observation will be crucial in our proof of Theorem 2*.

3. Proof of Theorem 2*

By Propositions 1 and 3 it is enough to prove that if the graph G^n is bipartite and all the degrees are at least $r=(100\,(K+m))^{k^2}n^{1/k}$ then G^n contains L_k, where K and m were defined in Lemma 3. Suppose $G^n\not\supseteq L_k$.

Fix a vertex u and denote by S_p the set of vertices having distance p from u. We shall prove by induction on p that

(a) $|S_{p-1}|<\frac{1}{2}|S_p|$ for $p\leqq k$, and

(b) $|S_p|\geqq \tilde{c}_p r^p$ if $p\leqq k$ and $\tilde{c}_p=\dfrac{1}{(100(K+m))^{p^2}}$.

The assertion is trivial for $p=1$. Assume that we have it for some $p<k$ and wish to prove it for $p+1$. For each vertex $w\in S_p$ a path P_w of length p can be fixed, joining w to u, and therefore meeting each level just once. Choosing these paths

one by one we may also easily achieve that *the union of these paths forms a tree* F. If we knew that each $w \in S_p$ is joined to at least $r/2$ vertices of S_{p+1}, then, applying Lemma 3, we would immediately obtain the more important assertion, namely, (*b*). However, it many occur that a small number of vertices of S_p are joined by many edges to S_{p-1} and by a small number of edges to S_{p+1}.

Let S^* be the set of vertices of S_p joined to S_{p+1} by less than $r/2$ edges. These vertices are joined to S_{p-1} by at least $r/2$. edges. Let $s_{p-1} := \left[\frac{1}{\log n} |S_{p-1}|\right]$. We show that $|S^*| < s_{p-1}$. Indeed, otherwise we can fix an $S' \subseteq S^*$ with s_{p-1} vertices and apply (14) with $U = S'$, $V = S_{p-1}$:

$$|S_{p-1}| > c'r \frac{1}{\log n} |S_{p-1}| > c'' \frac{\sqrt[k]{n}}{\log n} |S_{p-1}|,$$

which is a contradiction if n is sufficiently large. Put $\tilde{S}_p = S_p - S^*$. By (a) (applied with p) we have $|\tilde{S}_p| \geqq 1/2 |S_p|$, and each vertex of $\tilde{S}_p$ is joined to S_{p+1} by at least $r/2$ edges. Replacing the tree F by the corresponding subtree $\tilde{F}$, we may apply Lemma 3:

$$|S_{p+1}| \geqq c_p r |\tilde{S}_p| \geqq c_p \frac{r}{2} |S_p| > \tilde{c}_{p+1} r^{p+1}. \tag{20}$$

If n is sufficiently large, then (a) is trivial from (20).

As mentioned in our last Remark, if T is a tree, then the argument above is valid not only for $p+1 \leqq k-1$, but for $p+1 = k$ as well. However, this would give $|S_k| \geqq c_k r^k > n$, a contradiction. ∎

4. Sketch of the proof of Theorem 3

The above argument can easily be generalized to an arbitrary T, however, we have to fix a lower bound on r yielding the contradiction $|S_p| \geqq n$ already for $p = k-1$. A short calculation yields that $|S_1| \geqq r$,

$$|S_2| \geqq \tilde{\tilde{c}}_2 r r^\alpha = \tilde{c}_2 r^{1+\alpha}$$

$$|S_3| \geqq \tilde{\tilde{c}}_3 r (r^{1+\alpha})^\alpha = \tilde{\tilde{c}}_3 r^{1+\alpha+\alpha^2},$$

and, in general,

$$|S_p| \geqq \tilde{\tilde{c}}_p r^{1+\alpha+\alpha^2+\dots+\alpha^p}.$$

We get the desired contradiction, if $\tilde{\tilde{c}}_{k-1} r^{1+\alpha+\dots+\alpha^{k-1}} \geqq n$, which yields that if G^n contains no L_k, then for some large $\tilde{K}$,

$$e(G^n) \leqq \tilde{\tilde{K}} n^{1+\frac{1}{1+\alpha+\alpha^2+\dots+\alpha^{k-1}}}$$

and this is an equivalent form of Theorem 3. ∎

The method used in this paper could be called "blowing up" method, since the sets $|S_p|$ are forced to grow rapidly by the condition that G^n contains no L_k.

Remark. As we have mentioned, Theorem 3 can be improved. We shall return to the detailed discussion of the general case in another paper. Here we mention only, that (among others) one weak point in the proof of Theorem 3 is that we use $\sum a_i^\gamma \geqq (\sum a_i)^\gamma$, which is definitely weak if we have many a_i's and they have approximately the same size.

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